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LARGE DEFORMATIONS OF A LAMINATED COMPOSITE†

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Abstract—An approximate nonlinear theory is derived to describe the mechanical behavior of a laminated composite consisting of alternating layers of two homogeneous materials subjected to large deformations. The theory is based on two-term expansions of the motion across the thicknesses of the undeformed layers. The kinematics and the balance laws are formulated, and the constitutive equations are worked out for elastic behavior of the constitutive materials. The governing equations are subsequently written out in detail for the case of a small amplitude disturbance superimposed on a large static deformation. The latter system of equations is used to investigate the propagation of small amplitude time-harmonic waves in a prestressed laminated composite.

1. INTRODUCTION

THE states of stress and deformation in a layered medium can conceptually be analyzed by seeking solutions to the system of governing equations in each layer and by requiring these solutions to satisfy appropriate continuity conditions at the interfaces as well as prescribed conditions at the outer surfaces of the body. A rigorous analysis may, however, encounter rather serious difficulties, especially for large deformations. It is therefore of interest to investigate certain gross aspects of the mechanical behavior of a laminated composite by means of a homogeneous continuum model which takes into account, in an approximate manner, the lamellar structuring of the solid. For a layered medium consisting of homogeneous, isotropic, linearly elastic layers subjected to small deformations, a theory for a homogeneous continuum model was introduced in Refs. [1, 2]. The theory was extended in Ref. [3] to include effects of anisotropy, viscoelasticity and temperature variations. In the present paper we construct an approximate theory which can describe large deformations of a laminated elastic composite.

The system of governing equations for the homogeneous continuum model of the laminated medium is derived in two stages. The first stage of the derivation involves certain assumptions and operations within the discrete system of layers. In particular, it is assumed that the motions of the individual layers can be described by two-term expansions in the local coordinate normal to the layering of the undeformed body. The kinematic variables that are introduced in the expansions are defined at the midplanes of the layers only. Also in the discrete system of layers, balance equations of linear momentum and moment of momentum for the individual layers are obtained by integrating the local balance equations across the thicknesses of the undeformed layers. These integrations lead to the definitions of average stress tensors and couple-stress tensors which are again defined in

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discrete planes only. The stresses and couple stresses are subsequently related to the quantities describing the deformation through stress potentials which have also been obtained by integrating local stress potentials across the undeformed thicknesses. In the next stage of the derivation a transition is made from the system of discrete layers to the homogeneous continuum model. The transition is accomplished by defining fields for the kinematical and dynamical variables that are continuous in the coordinate normal to the layering. In prescribed discrete planes parallel to the layering the field variables assume the same values as the variables that were defined in the discrete system of layers. The resulting system of nonlinear field equations, consisting of balance equations, constitutive relations and a constraint condition, bears a close resemblance to equations defining a nonlinear theory of elasticity with microstructure.

The approximate theory of the type presented in this paper is useful if the characteristic length of the variation of the deformation is larger than the characteristic length of the structuring. For small deformations the continuum model was used in Ref. [1] to study the propagation of plane harmonic waves. For waves propagating in the direction of the layering and normal to the direction of the layering, it was shown that the approximate phase velocity vs. wavenumber curves show good agreement with exact curves. The equations for large deformations that are derived in this paper are employed to study the propagation of small amplitude time-harmonic waves superimposed on a large static deformation. The dispersion relations are derived. It is shown that the results are very similar to those of Ref. [1], except that the coefficients depend not only on the structuring, but also on the large static deformation.

NOTATION

Throughout this paper we use standard Cartesian tensor notation. The positions of material particles before and after deformation are referred to Cartesian coordinate systems. Upper case italic subscripts assume the values 1, 2, 3 and indicate tensors in the Lagrangian system X_K. Lower case italic subscripts assume the values 1, 2, 3 and specify tensors in the Eulerian system x_l. Greek subscripts assume the values 1, 3 and are referred to the X_1, X_3 components of the Lagrangian system. Superscripts in parentheses indicate whether a quantity belongs to a reinforcing layer or a matrix layer; they are not tensor indices.

2. DEFORMATION

We consider a medium which before deformation consists of alternating plane layers of two homogeneous materials. It is specified that the field variables and the material parameters in the material whose resistance to deformation is larger (the reinforcing layers) are denoted by superscripts and subscripts f (fiber). The corresponding quantities in the other layers (the matrix layers) are denoted by superscripts and subscripts m. In the undeformed body we choose the direction X_2 perpendicular to the layering, see Fig. 1, and we consider the motion of the kth reinforcing layer and the kth matrix layer whose midplane positions in the undeformed state are defined by $X_2^{(fk)}$ and $X_2^{(mk)}$, respectively. If the length characterizing the variation of the deformation is large compared to the thicknesses of the layers, the motion of the kth pair of layers can in first approximation

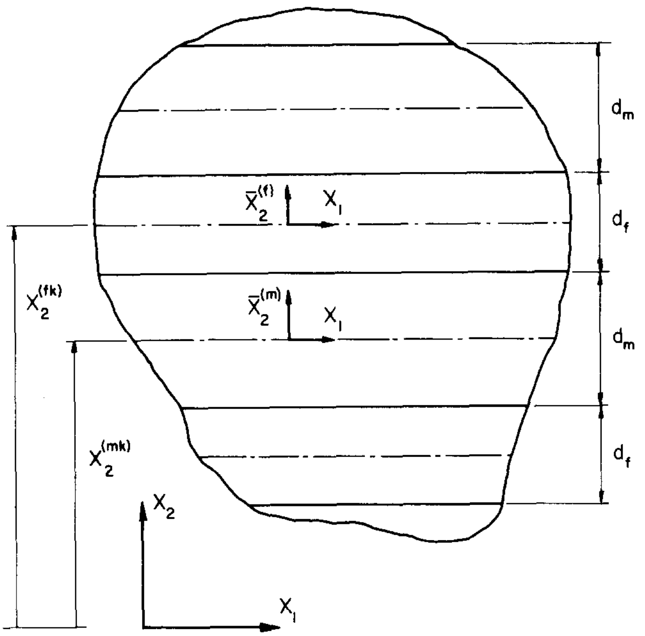

FIG. 1. Laminated medium in the undeformed state.

be described by

$$x_l^{(fk)} = \bar{x}_l^{(fk)}(X_1, X_2^{(fk)}, X_3, t) + \bar{X}_2^{(f)}\psi_{2l}^{(fk)}(X_1, X_2^{(fk)}, X_3, t) \tag{2.1}$$

$$x_l^{(mk)} = \bar{x}_l^{(mk)}(X_1, X_2^{(mk)}, X_3, t) + \bar{X}_2^{(m)}\psi_{2l}^{(mk)}(X_1, X_2^{(mk)}, X_3, t), \tag{2.2}$$

where $\bar{x}_l^{(fk)}$ and $\bar{x}_l^{(mk)}$ represent the motions of the midplanes of the undeformed layers. In equations (2.1) and (2.2), $\psi_{21}^{(fk)}$ and $\psi_{23}^{(fk)}$ describe antisymmetric thickness shear motions, and $\psi_{22}^{(fk)}$ describes symmetric thickness stretch motion of the kth reinforcing layer. Similar interpretations hold for $\psi_{2l}^{(mk)}$. The expansions for the components of the motion, (2.1) and (2.2), are in terms of local coordinates $\bar{X}_2^{(f)}$ and $\bar{X}_2^{(m)}$ in the undeformed body, see Fig. 1.

The continuity of the position at the interface of the kth pair of layers requires that $\bar{x}_l^{(fk)}, \bar{x}_l^{(mk)}, \psi_{2l}^{(fk)}$, and $\psi_{2l}^{(mk)}$ satisfy

$$\bar{x}_l^{(fk)}(X_1, X_2^{(fk)}, X_3, t) - \bar{x}_l^{(mk)}(X_1, X_2^{(mk)}, X_3, t) = \tfrac{1}{2}d_f\psi_{2l}^{(fk)}(X_1, X_2^{(fk)}, X_3, t) + \tfrac{1}{2}d_m\psi_{2l}^{(mk)}(X_1, X_2^{(mk)}, X_3, t). \tag{2.3}$$

In the system of discrete layers the state of deformation is now described by the field variables $\bar{x}_l^{(fk)}$, $\psi_{2l}^{(fk)}$ and $\bar{x}_l^{(mk)}$, $\psi_{2l}^{(mk)}$, which are defined only at the midplanes of the undeformed reinforcing and matrix layers, respectively. We now proceed in the same manner as in the linear theories of Refs. [1–3], i.e. we construct a homogeneous continuum model for the laminated medium by considering the above field variables as continuous functions of X_2 whose values at $X_2 = X_2^{(fk)}$ and $X_2 = X_2^{(mk)}$ coincide with the actual values at the midplanes of the undeformed layers. The transition to the continuous field is indicated by writing $\bar{x}_l^{(f)}(X_K, t)$ instead of $\bar{x}_l^{(fk)}(X_1, X_2^{(fk)}, X_3, t)$, etc. After this transition the deformation is described by the field variables $\bar{x}_l^{(f)}$, $\bar{x}_l^{(m)}$, $\psi_{2l}^{(f)}$, and $\psi_{2l}^{(m)}$. The number of variables is subsequently reduced by noting that $\bar{x}_l^{(f)}$ and $\bar{x}_l^{(m)}$ should be considered as representing the same function at different locations. We thus replace $\bar{x}_l^{(f)}$ and $\bar{x}_l^{(m)}$ by $\bar{x}_l(X_K, t)$, and henceforth we refer to this quantity as the "gross motion." By noting that $X_2^{(mk)} = X_2^{(fk)} - \frac{1}{2}(d_f + d_m)$, and by assuming that the thicknesses of the layers are sufficiently small, the difference relations (2.3) can now be replaced by the differential relations

$$\partial_2 \bar{x}_l(X_K, t) = \eta \psi_{2l}^{(f)}(X_K, t) + (1-\eta)\psi_{2l}^{(m)}(X_K, t), \tag{2.4}$$

where

$$\partial_K = \partial(\ )/\partial X_K, \tag{2.5}$$

and η is defined as

$$\eta = d_f/(d_f + d_m). \tag{2.6}$$

Equation (2.4) is a constraint condition which holds anywhere in the continuum. The passage from (2.3) to (2.4) involves a limiting process which is formally justifiable in the limits $d_f \to 0$ and $d_m \to 0$ but keeping η constant.

3. BALANCE OF MOMENTUM

To derive the dynamical balance laws for the homogeneous continuum model we first consider the balance of linear momentum within each discrete layer. For the kth reinforcing and matrix layers we have

$$t_{nl,n}^{(fk)} + \rho^{(fk)} f_l^{(fk)} = \rho^{(fk)} a_l^{(fk)} \tag{3.1}$$

$$t_{nl,n}^{(mk)} + \rho^{(mk)} f_l^{(mk)} = \rho^{(mk)} a_l^{(mk)}, \tag{3.2}$$

where we have used the notation

$$(\ )_{,n} = \partial(\ )/\partial x_n, \tag{3.3}$$

where x_n are the Eulerian coordinates, and where $t_{nl}^{(fk)}$ and $t_{nl}^{(mk)}$ are the Cauchy stress tensors in the kth reinforcing layer and the kth matrix layer, respectively. Also, $\rho^{(fk)}$ and $\rho^{(mk)}$ are the mass densities, $f_l^{(fk)}$ and $f_l^{(mk)}$ the body forces, and $a_l^{(fk)}$ and $a_l^{(mk)}$ are the accelerations.

It is convenient to introduce the Piola stress tensor which is defined as [4, 5]

$$T_{Kn}^{(fk)} = J^{(fk)} X_{K,l}^{(fk)} t_{ln}^{(fk)} \tag{3.4}$$

$$T_{Kn}^{(mk)} = J^{(mk)} X_{K,l}^{(mk)} t_{ln}^{(mk)}, \tag{3.5}$$

wherein

$$J^{(fk)} = \det[\partial_K x_l^{(fk)}], \qquad J^{(mk)} = \det[\partial_K x_l^{(mk)}]. \tag{3.6a, b}$$

We also introduce the densities in the undeformed state, ρ_f and ρ_m, which may be written in the form

$$\rho_f = \rho^{(fk)} J^{(fk)} \quad \text{and} \quad \rho_m = \rho^{(mk)} J^{(mk)}. \tag{3.7a, b}$$

In this paper we assume that the undeformed layers are homogeneous, i.e. ρ_f and ρ_m are constants. Upon substitution of equations (3.4), (3.5) and (3.7a, b), the balance laws (3.1) and (3.2) can be written as (see, e.g. [4, p. 109] or [5, p. 553])

$$\partial_K T_{Kl}^{(fk)} + \rho_f f_l^{(fk)} = \rho_f \ddot{x}_l^{(fk)} \tag{3.8}$$

$$\partial_K T_{Kl}^{(mk)} + \rho_m f_l^{(mk)} = \rho_m \ddot{x}_l^{(mk)}. \tag{3.9}$$

In equations (3.8) and (3.9) the field variables are considered as functions of the undeformed local coordinates in each layer.

A rigorous solution for the stress distribution and the motion in a layered medium would be obtained by solving (3.8) and (3.9) in each layer and by requiring that appropriate conditions be satisfied at the interfaces of the layers and at the outer surfaces of the body. If a length parameter characterizing the variation of the deformation is, however, large compared to a length parameter characterizing the structuring, some interesting information on certain gross dynamic quantities can be obtained from a homogeneous continuum model. Such a model can be constructed in the same manner as for the case of small deformations, which was discussed in Refs. [1–3].

To derive equations representing balance of linear momentum for the continuum model we first integrate (3.8) and (3.9) in the undeformed frame over their respective layer thicknesses and then we add the results to obtain

$$\partial_1 \bar{T}_{1l}^{(k)} + \partial_3 \bar{T}_{3l}^{(k)} + \frac{1}{d_f + d_m}\{[T_{2l}^{(fk)}]_{\bar{X}_2^{(f)} = \frac{1}{2}d_f} - [T_{2l}^{(mk)}]_{\bar{X}_2^{(m)} = -\frac{1}{2}d_m}\} + \rho \bar{f}_l^{(k)} = \rho \bar{a}_l^{(k)}, \tag{3.10}$$

where we have used the conditions of continuity of the stresses at the interface between the layers, and where

$$(d_f + d_m)\bar{T}_{\alpha l}^{(k)} = \mathrm{Int}^{(fk)}[T_{\alpha l}^{(fk)}] + \mathrm{Int}^{(mk)}[T_{\alpha l}^{(mk)}] \tag{3.11}$$

$$\rho(d_f + d_m)\bar{f}_l^{(k)} = \mathrm{Int}^{(fk)}[\rho_f f_l^{(fk)}] + \mathrm{Int}^{(mk)}[\rho_m f_l^{(mk)}] \tag{3.12}$$

$$\rho(d_f + d_m)\bar{a}_l^{(k)} = \mathrm{Int}^{(fk)}[\rho_f \ddot{x}_l^{(fk)}] + \mathrm{Int}^{(mk)}[\rho_m \ddot{x}_l^{(fk)}]. \tag{3.13}$$

In equation (3.11), $\alpha = 1, 3$. Also,

$$\rho = \eta\rho_f + (1 - \eta)\rho_m. \tag{3.14}$$

In (3.11–3.13) we have employed the following notation to indicate integration of a function $g^{(fk)}(X_1, X_2^{(fk)}, \bar{X}_2^{(f)}, X_3, t)$ in the undeformed frame over the thickness of the reinforcing layer:

$$\mathrm{Int}^{(fk)}[g^{(fk)}] = \int_{-\frac{1}{2}d_f}^{\frac{1}{2}d_f} g^{(fk)}(X_1, X_2^{(fk)}, \bar{X}_2^{(f)}, X_3, t)\, \mathrm{d}\bar{X}_2^{(f)}. \tag{3.15}$$

We have also used

$$\mathrm{Int}^{(mk)}[g^{(mk)}] = \int_{-\frac{1}{2}d_m}^{\frac{1}{2}d_m} g^{(mk)}(X_1, X_2^{(mk)}, \bar{X}_2^{(m)}, X_3, t)\, \mathrm{d}\bar{X}_2^{(m)}. \tag{3.16}$$

The balance equations (3.10) are valid for each discrete pair of layers. As in Ref. [3], the transition from the system of discrete layers to the homogeneous continuum model is achieved by considering $\bar{T}_{\alpha l}^{(k)}(\alpha = 1, 3)$ as the value at some position X_2, within the kth pair of layers in the undeformed body, of a field $\bar{T}_{\alpha l}(X_K, t)$ which is continuous in X_2. The same argument is used for the averaged body force $\bar{f}_l^{(k)}$. By employing the field $\bar{x}_l(X_K, t)$

which was introduced earlier, and neglecting the difference between the time derivatives at $X_2 = X_2^{(fk)}$ and $X_2 = X_2^{(mk)}$, we can, furthermore, replace $\bar{a}_l^{(k)}$ by the field $\bar{a}_l(X_K, t)$,

$$\bar{a}_l(X_K, t) = \rho\ddot{x}_l(X_K, t). \tag{3.17}$$

The remaining term in (3.10) is the difference between the interface stresses at the top and bottom of the kth pair of layers. If we introduce a continuous function $\Sigma_{2l}(X_K, t)$ whose values coincide with the interface stresses at the interfaces of the layers, then we can approximate the difference in (3.10) by the derivative of Σ_{2l} with respect to X_2, i.e.

$$\frac{1}{d_f+d_m}\{[T_{2l}^{(fk)}]_{\bar{X}_2^{(f)}=\frac{1}{2}d_f} - [\bar{T}_{2l}^{(mk)}]_{\bar{X}_2^{(m)}=-\frac{1}{2}d_m}\} \simeq \partial_2\Sigma_{2l}. \tag{3.18}$$

The balance laws (3.10) thus reduce to

$$\partial_\alpha\bar{T}_{\alpha l}+\partial_2\Sigma_{2l}+\rho\bar{f}_l = \rho\ddot{x}_l, \tag{3.19}$$

where $\alpha = 1, 3$.

To deduce the balance laws for the moment of momentum of the continuum model we first multiply (3.8) by $\bar{X}_2^{(f)}$ and (3.9) by $\bar{X}_2^{(m)}$, and then integrate over the respective layer thicknesses. For the kth reinforcing layer we obtain

$$\partial_1 M_{12l}^{(fk)}+\partial_3 M_{32l}^{(fk)}+\frac{1}{d_f}\mathrm{Int}^{(fk)}\left[\bar{X}_2^{(f)}\frac{\partial T_{2l}^{(fk)}}{\partial\bar{X}_2^{(f)}}\right]+\rho_f l_{2l}^{(fk)} = \omega_l^{(fk)}, \tag{3.20}$$

where

$$d_f M_{\alpha 2l}^{(fk)} = \mathrm{Int}^{(fk)}[\bar{X}_2^{(f)}T_{\alpha l}^{(fk)}], \qquad \alpha = 1, 3 \tag{3.21}$$

$$d_f\rho_f l_{2l}^{(fk)} = \mathrm{Int}^{(fk)}[\bar{X}_2^{(f)}\rho_f f_l^{(fk)}] \tag{3.22}$$

$$d_f\omega_l^{(fk)} = \mathrm{Int}^{(fk)}[\bar{X}_2^{(f)}\rho_f\ddot{x}_l^{(fk)}]. \tag{3.23}$$

The integral appearing in (3.20) can be rewritten as

$$\frac{1}{d_f}\mathrm{Int}^{(fk)}\left[\bar{X}_2^{(f)}\frac{\partial T_{2l}^{(fk)}}{\partial\bar{X}_2^{(f)}}\right] = \tfrac{1}{2}\{[T_{2l}^{(fk)}]_{\bar{X}_2^{(f)}=\frac{1}{2}d_f}+[T_{2l}^{(fk)}]_{\bar{X}_2^{(f)}=-\frac{1}{2}d_f}\}-\bar{T}_{2l}^{(fk)}, \tag{3.24}$$

where we have introduced

$$d_f\bar{T}_{2l}^{(fk)} = \mathrm{Int}^{(fk)}[T_{2l}^{(fk)}]. \tag{3.25}$$

The transition from the system of discrete layers to the homogeneous continuum model is effected in exactly the same manner as for the equations of linear momentum. Thus we introduce the field quantities $M_{\alpha 2l}^{(f)}(X_K, t)$, $\bar{T}_{2l}^{(f)}(X_K, t)$, $l_{2l}^{(f)}(X_K, t)$, which coincide with $M_{\alpha 2l}^{(fk)}$, $\bar{T}_{2l}^{(fk)}$, $l_{2l}^{(fk)}$, at positions within the discrete layers. Using (2.1), $\omega_l^{(fk)}$ becomes

$$\omega_l^{(fk)} = I_f\ddot{\psi}_{2l}^{(fk)}, \quad \text{where } I_f = \tfrac{1}{12}\rho_f d_f^2. \tag{3.26}$$

In the continuum model, $\omega_l^{(fk)}$ is represented by

$$\omega_l^{(f)} = I_f\ddot{\psi}_{2l}^{(f)}. \tag{3.27}$$

Finally, in the first approximation we can write the term in brackets in (3.24) as

$$\tfrac{1}{2}\{[T_{2l}^{(fk)}]_{\bar{X}_2^{(f)}=\frac{1}{2}d_f}+[T_{2l}^{(fk)}]_{\bar{X}_2^{(f)}=-\frac{1}{2}d_f}\} \simeq \Sigma_{2l}(X_K, t). \tag{3.28}$$

The same reasoning can be repeated for the kth matrix layer. Collecting the results, we obtain the following system of balance laws of moment of momentum:

$$\partial_\alpha M_{\alpha 2l}^{(f)}+\Sigma_{2l}-\bar{T}_{2l}^{(f)}+\rho_f l_{2l}^{(f)} = I_f\ddot{\psi}_{2l}^{(f)} \tag{3.29}$$

$$\partial_\alpha M_{\alpha 2l}^{(m)}+\Sigma_{2l}-\bar{T}_{2l}^{(m)}+\rho_m l_{2l}^{(m)} = I_m\ddot{\psi}_{2l}^{(m)}. \tag{3.30}$$

Equations (3.19), (3.29) and (3.30) constitute the basic balance laws for the continuum model of a laminated composite. It should be noted that the interface stress Σ_{2l} is considered an unknown quantity which is to be determined from the balance laws. The boundary conditions corresponding to the balance laws (3.19), (3.29) and (3.30) are

$$T_{(\mathbf{N})l} = \bar{T}_{\alpha l}N_\alpha+\Sigma_{2l}N_2 \tag{3.31}$$

$$M_{(\mathbf{N})2l}^{(f)} = M_{\alpha 2l}^{(f)}N_\alpha \tag{3.32}$$

$$M_{(\mathbf{N})2l}^{(m)} = M_{\alpha 2l}^{(m)}N_\alpha, \tag{3.33}$$

where $\mathbf{N}$ is the normal to the boundary surface before deformation. These conditions are discussed in more detail in Ref. [3].

4. CONSTITUTIVE EQUATIONS FOR ELASTIC MATERIALS

The system of governing equations of the homogeneous continuum model for a layered composite is completed by specifying the mechanical response of the materials of the layers. We consider a layered composite consisting of elastic layers. The constitutive equations for the kth pair of reinforcing and matrix layers may then be written as

$$T_{Kl}^{(fk)} = \rho_f\frac{\partial F^{(fk)}}{\partial(\partial_K x_l^{(fk)})}, \qquad F^{(fk)} = F_f(\partial_K x_l^{(fk)}) \tag{4.1}$$

$$T_{Kl}^{(mk)} = \rho_m\frac{\partial F^{(mk)}}{\partial(\partial_K x_l^{(mk)})}, \qquad F^{(mk)} = F_m(\partial_K x_l^{(mk)}), \tag{4.2}$$

respectively, where F_f and F_m are the elastic stress potentials of the reinforcing material and the matrix material, respectively. We now introduce the elastic potential $\bar{F}$ of the layered composite as:

$$\begin{aligned}\rho\bar{F}(\partial_\alpha\bar{x}_l, \partial_\alpha\psi_{2l}^{(f)}, \partial_\alpha\psi_{2l}^{(m)}, \psi_{2l}^{(f)}, \psi_{2l}^{(m)}) &= \frac{1}{d_f+d_m}\int_{-\frac{1}{2}d_f}^{\frac{1}{2}d_f}\rho_f F_f(\partial_\alpha\bar{x}_l+\bar{X}_2^{(f)}\partial_\alpha\psi_{2l}^{(f)}, \psi_{2l}^{(f)})\,\mathrm{d}\bar{X}_2^{(f)}\\ &\quad+\frac{1}{d_f+d_m}\int_{-\frac{1}{2}d_m}^{\frac{1}{2}d_m}\rho_m F_m(\partial_\alpha\bar{x}_l+\bar{X}_2^{(m)}\partial_\alpha\psi_{2l}^{(m)}, \psi_{2l}^{(m)})\,\mathrm{d}\bar{X}_2^{(m)}.\end{aligned} \tag{4.3}$$

In (4.3) we have used the form (2.1) of the motion and passed to the continuum model in the usual manner. Using (4.1)–(4.3) and passing to the continuum model, we obtain the following constitutive equations for a layered elastic composite:

$$\bar{T}_{\alpha l} = \rho \frac{\partial \bar{F}}{\partial(\partial_\alpha \bar{x}_l)} \tag{4.4}$$

$$\eta M^{(f)}_{\alpha 2l} = \rho \frac{\partial \bar{F}}{\partial(\partial_\alpha \psi^{(f)}_{2l})} \tag{4.5}$$

$$(1-\eta) M^{(m)}_{\alpha 2l} = \rho \frac{\partial \bar{F}}{\partial(\partial_\alpha \psi^{(m)}_{2l})} \tag{4.6}$$

$$\eta \bar{T}^{(f)}_{2l} = \rho \frac{\partial \bar{F}}{\partial \psi^{(f)}_{2l}} \tag{4.7}$$

$$(1-\eta) \bar{T}^{(m)}_{2l} = \rho \frac{\partial \bar{F}}{\partial \psi^{(m)}_{2l}}, \tag{4.8}$$

where η is defined by equation (2.6).

Equations (4.4)–(4.8) are the constitutive equations, according to the continuum model, of a nonlinear laminated elastic material. Equations (2.4), (3.19), (3.29)–(3.30) along with the constitutive equations (4.4)–(4.8) form a complete system of nonlinear differential equations for the gross motion $\bar{x}_l$, the local motions $\psi^{(f)}_{2l}$, $\psi^{(m)}_{2l}$ and the interface stress vector Σ_{2l}.

By substituting the constitutive equations (4.4)–(4.8) into the balance laws (3.19), (3.29)–(3.30) and neglecting body forces and body couples, we obtain the following set of governing equations:

$$\frac{\partial \bar{T}_{\alpha l}}{\partial(\partial_\beta \bar{x}_n)} \partial_\alpha \partial_\beta \bar{x}_n + \frac{\partial \bar{T}_{\alpha l}}{\partial(\partial_\beta \psi^{(f)}_{2n})} \partial_\alpha \partial_\beta \psi^{(f)}_{2n} + \frac{\partial \bar{T}_{\alpha l}}{\partial(\partial_\beta \psi^{(m)}_{2n})} \partial_\alpha \partial_\beta \psi^{(m)}_{2n} + \frac{\partial \bar{T}_{\alpha l}}{\partial \psi^{(f)}_{2n}} \partial_\alpha \psi^{(f)}_{2n}$$

$$+ \frac{\partial \bar{T}_{\alpha l}}{\partial \psi^{(m)}_{2n}} \partial_\alpha \psi^{(m)}_{2n} + \partial_2 \Sigma_{2l} = \rho \ddot{\bar{x}}_l \tag{4.9}$$

$$\frac{\partial M^{(f)}_{\alpha 2l}}{\partial(\partial_\beta \bar{x}_n)} \partial_\alpha \partial_\beta \bar{x}_n + \frac{\partial M^{(f)}_{\alpha 2l}}{\partial(\partial_\beta \psi^{(f)}_{2n})} \partial_\alpha \partial_\beta \psi^{(f)}_{2n} + \frac{\partial M^{(f)}_{\alpha 2l}}{\partial \psi^{(f)}_{2n}} \partial_\alpha \psi^{(f)}_{2n} + \Sigma_{2l} - \bar{T}^{(f)}_{2l} = I_f \ddot{\psi}^{(f)}_{2l} \tag{4.10}$$

$$\frac{\partial M^{(m)}_{\alpha 2l}}{\partial(\partial_\beta \bar{x}_n)} \partial_\alpha \partial_\beta \bar{x}_n + \frac{\partial M^{(m)}_{\alpha 2l}}{\partial(\partial_\beta \psi^{(m)}_{2n})} \partial_\alpha \partial_\beta \psi^{(m)}_{2n} + \frac{\partial M^{(m)}_{\alpha 2l}}{\partial \psi^{(m)}_{2n}} \partial_\alpha \psi^{(m)}_{2n} + \Sigma_{2l} - \bar{T}^{(m)}_{2l} = I_m \ddot{\psi}^{(m)}_{2l}. \tag{4.11}$$

For completeness we also list the constraint condition (2.4)

$$\partial_2 \bar{x}_l = \eta \psi^{(f)}_{2l} + (1-\eta) \psi^{(m)}_{2l}. \tag{4.12}$$

5. SMALL DEFORMATIONS SUPERIMPOSED ON A LARGE UNIFORM DEFORMATION

In this section we derive the governing equations for a disturbance of small amplitude superimposed on a large static deformation. The static deformation is described by

$$\bar{x}_l^0 = A_{\alpha l} X_\alpha + A_{2l} X_2 + b_l$$
$$\psi_{2l}^{(f)0} = B_{2l}^{(f)} \qquad (5.1)$$
$$\psi_{2l}^{(m)0} = B_{2l}^{(m)},$$

where $A_{\alpha l}$, A_{2l}, $B_{2l}^{(f)}$, $B_{2l}^{(m)}$ and b_l are constants. The constraint equation (2.4) requires that

$$A_{2l} = \eta B_{2l}^{(f)} + (1-\eta) B_{2l}^{(m)}. \qquad (5.2)$$

From the balance equations (3.29) and (3.30) we conclude that

$$\Sigma_{2l}^0 = \bar{T}_{2l}^{(f)}(A_{\alpha n}, B_{2n}^{(f)}) = \bar{T}_{2l}^{(m)}(A_{\alpha n}, B_{2n}^{(m)}). \qquad (5.3)$$

Equation (5.3) can be considered as an equation determining $B_{2n}^{(f)}$ in terms of $A_{\alpha n}$ and $B_{2n}^{(m)}$, or conversely as determining $B_{2n}^{(m)}$ in terms of $A_{\alpha n}$ and $B_{2n}^{(f)}$.†

We consider motions defined by

$$\bar{x}_l = \bar{x}_l^0 + \bar{u}_l \qquad (5.4)$$

$$\psi_{2l}^{(f)} = \psi_{2l}^{(f)0} + \hat{\psi}_{2l}^{(f)} \qquad (5.5)$$

$$\psi_{2l}^{(m)} = \psi_{2l}^{(m)0} + \hat{\psi}_{2l}^{(m)} \qquad (5.6)$$

$$\Sigma_{2l} = \Sigma_{2l}^0 + \hat{\Sigma}_{2l}, \qquad (5.7)$$

where $\bar{u}_l$, $\hat{\psi}_{2l}^{(f)}$, $\hat{\psi}_{2l}^{(m)}$ and $\hat{\Sigma}_{2l}$ and their first order derivatives are small quantities compared to $\bar{x}_l^0$, etc. To within linear terms in $\bar{u}_l$, $\hat{\psi}_{2l}^{(f)}$, $\hat{\psi}_{2l}^{(m)}$ and $\hat{\Sigma}_{2l}$ the system (4.9–4.12) reduces to

$$\bar{E}_{\alpha\beta nl}\partial_\alpha\partial_\beta\bar{u}_n + \eta D_{\alpha nl}^{(f)}\partial_\alpha\hat{\psi}_{2n}^{(f)} + (1-\eta)D_{\alpha nl}^{(m)}\partial_\alpha\hat{\psi}_{2n}^{(m)} + \partial_2\hat{\Sigma}_{2l} = \rho\ddot{\bar{u}}_l \qquad (5.8)$$

$$E_{\alpha\beta nl}^{(f)}\partial_\alpha\partial_\beta\hat{\psi}_{2n}^{(f)} + \hat{\Sigma}_{2l} - D_{\alpha ln}^{(f)}\partial_\alpha\bar{u}_n - H_{nl}^{(f)}\hat{\psi}_{2n}^{(f)} = I_f\ddot{\hat{\psi}}_{2l}^{(f)} \qquad (5.9)$$

$$E_{\alpha\beta nl}^{(m)}\partial_\alpha\partial_\beta\hat{\psi}_{2n}^{(m)} + \hat{\Sigma}_{2l} - D_{\alpha ln}^{(m)}\partial_\alpha\bar{u}_n - H_{nl}^{(m)}\hat{\psi}_{2n}^{(m)} = I_m\ddot{\hat{\psi}}_{2l}^{(m)} \qquad (5.10)$$

$$\partial_2\bar{u}_l = \eta\hat{\psi}_{2l}^{(f)} + (1-\eta)\hat{\psi}_{2l}^{(m)}, \qquad (5.11)$$

† The solution which follows from (5.1)–(5.3) is identical to the exact solution obtained by considering the discrete layers, as shown in the Appendix.

where

$$
\begin{aligned}
\bar{E}_{\alpha\beta nl} &= \rho_f \eta \frac{\partial^2 F_f^0}{\partial A_{\alpha l} \partial A_{\beta n}} + \rho_m (1-\eta) \frac{\partial^2 F_m^0}{\partial A_{\alpha l} \partial A_{\beta n}} \\
D_{\alpha nl}^{(f)} &= \rho_f \frac{\partial^2 F_f^0}{\partial A_{\alpha l} \partial B_{2n}^{(f)}} \\
D_{\alpha nl}^{(m)} &= \rho_m \frac{\partial^2 F_m^0}{\partial A_{\alpha l} \partial B_{2n}^{(m)}} \\
E_{\alpha\beta nl}^{(f)} &= I_f \frac{\partial^2 F_f^0}{\partial A_{\alpha l} \partial A_{\beta n}} \\
E_{\alpha\beta nl}^{(m)} &= I_m \frac{\partial^2 F_m^0}{\partial A_{\alpha l} \partial A_{\beta n}} \\
H_{nl}^{(f)} &= \rho_f \frac{\partial^2 F_f^0}{\partial B_{2n}^{(f)} \partial B_{2l}^{(f)}} \\
H_{nl}^{(m)} &= \rho_m \frac{\partial^2 F_m^0}{\partial B_{2n}^{(m)} \partial B_{2l}^{(m)}},
\end{aligned}
\tag{5.12}
$$

where F_f^0 and F_m^0 are defined as

$$F_f^0 = F_f(A_{\alpha l}, B_{2l}^{(f)}) \tag{5.13}$$

$$F_m^0 = F_m(A_{\alpha l}, B_{2l}^{(m)}). \tag{5.14}$$

In deriving (5.8)–(5.10) we have used (4.3) to show that in the uniform state defined by (5.1)

$$\left.\frac{\partial^2 \bar{F}}{\partial(\partial_\alpha \bar{x}_l)\partial(\partial_\beta \psi_{2n}^{(f)})}\right|_0 = \left.\frac{\partial^2 \bar{F}}{\partial(\partial_\alpha \bar{x}_l)\partial(\partial_\beta \psi_{2n}^{(m)})}\right|_0 = \left.\frac{\partial^2 \bar{F}}{\partial(\partial_\alpha \psi_{2l}^{(f)})\partial \psi_{2n}^{(f)}}\right|_0 = \left.\frac{\partial^2 \bar{F}}{\partial(\partial_\alpha \psi_{2l}^{(m)})\partial \psi_{2n}^{(m)}}\right|_0 = 0. \tag{5.15}$$

The system of equations (5.8)–(5.11) are the governing dynamical equations, according to the continuum model, for a small deformation superimposed on a large uniform static deformation. If we assume that the materials constituting the lamina are isotropic and that the large deformation is such that one of the principal directions is normal to the layers, we can choose the x_l^0 system as

$$
\begin{aligned}
\bar{x}_1^0 &= \lambda_1 X_1 \\
\bar{x}_2^0 &= \lambda_2 X_2 \\
\bar{x}_3^0 &= \lambda_3 X_3.
\end{aligned}
\tag{5.16}
$$

We also assume that the local motions are of the form

$$
\begin{aligned}
\psi_{2l}^{(f)0} &= \lambda_2^{(f)} \delta_{2l} \\
\psi_{2l}^{(m)0} &= \lambda_2^{(m)} \delta_{2l},
\end{aligned}
\tag{5.17}
$$

where

$$\lambda_2 = \eta \lambda_2^{(f)} + (1-\eta)\lambda_2^{(m)}. \tag{5.18}$$

For an isotropic material, the stress potential is a function of the strain invariants,

$$F_f^0 = F_f^0(\mathrm{I}_f, \mathrm{II}_f, \mathrm{III}_f)$$
$$F_m^0 = F_m^0(\mathrm{I}_m, \mathrm{II}_m, \mathrm{III}_m). \tag{5.19}$$

In equation (5.19),

$$\begin{aligned} \mathrm{I}_f &= \mathrm{trace}\,(\mathbf{C}^{(f)}) = (\lambda_1)^2 + (\lambda_2^{(f)})^2 + (\lambda_3)^2 \\ \mathrm{II}_f &= \mathrm{trace}\,(\mathbf{C}^{(f)^2}) = (\lambda_1)^4 + (\lambda_2^{(f)})^4 + (\lambda_3)^4 \\ \mathrm{III}_f &= \mathrm{trace}\,(\mathbf{C}^{(f)^3}) = (\lambda_1)^6 + (\lambda_2^{(f)})^6 + (\lambda_3)^6, \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} C_{\alpha\beta}^{(f)} &= A_{\alpha l} A_{\beta l} \\ C_{\alpha 2}^{(f)} &= C_{2\alpha}^{(f)} = A_{\alpha l} B_{2l}^{(f)} \\ C_{22}^{(f)} &= B_{2l}^{(f)} B_{2l}^{(f)}. \end{aligned} \tag{5.21}$$

Similar expressions can be written for I_m, II_m, III_m and $C_{KL}^{(m)}$.

For convenience we introduce the following quantities which are related to the large static deformation:

(a) the densities of the statically deformed state

$$\begin{aligned} \rho_f^0 &= \frac{\rho_f}{\lambda_1 \lambda_3 \lambda_2^{(f)}} \\ \rho_m^0 &= \frac{\rho_m}{\lambda_1 \lambda_3 \lambda_2^{(m)}} \\ \rho^0 &= \frac{\rho}{\lambda_1 \lambda_2 \lambda_3} \end{aligned} \tag{5.22}$$

(b) the deformed layer thicknesses and their ratios

$$\begin{aligned} d_f^0 &= d_f \lambda_2^{(f)} \\ d_m^0 &= d_m \lambda_2^{(m)} \\ \eta^0 &= \frac{d_f^0}{d_f^0 + d_m^0} = \eta \frac{\lambda_2^{(f)}}{\lambda_2} \\ 1 - \eta^0 &= (1-\eta)\frac{\lambda_2^{(m)}}{\lambda_2}, \end{aligned} \tag{5.23}$$

where

$$\rho^0 = \rho_f^0 \eta^0 + \rho_m^0 (1-\eta^0) \tag{5.24}$$

(c) the principal stresses in the direction of the layering

$$\begin{aligned} t_\alpha^{(f)} &= \rho_f^0 \lambda_\alpha \frac{\partial F_f^0}{\partial \lambda_\alpha} \\ t_\alpha^{(m)} &= \rho_m^0 \lambda_\alpha \frac{\partial F_m^0}{\partial \lambda_\alpha}, \end{aligned} \qquad (\text{no sum on } \alpha) \tag{5.25}$$

and those in the direction normal to the layering

$$t_2 = \rho_f^0 \lambda_2^{(f)} \frac{\partial F_f^0}{\partial \lambda_2^{(f)}} = \rho_m^0 \lambda_2^{(m)} \frac{\partial F_m^0}{\partial \lambda_2^{(m)}}. \tag{5.26}$$

[Note that this is equation (5.3) written in the deformed body.] We also define the average principal stresses

$$\bar{t}_\alpha = \eta^0 t_\alpha^{(f)} + (1-\eta^0) t_\alpha^{(m)}. \tag{5.27}$$

By employing (5.16–5.27), the system (5.8–5.11) reduces to

$$\rho^0 \bar{\gamma}_{11} \partial_{\bar{1}\bar{1}} \bar{u}_1 + \rho^0 \bar{\gamma}_{31} \partial_{\bar{3}\bar{3}} \partial \bar{u}_1 + \rho^0 \bar{\alpha} \partial_{\bar{1}\bar{3}} \bar{u}_3 + \rho^0 \eta^0 \beta_{21}^{(f)} \partial_{\bar{1}} \bar{\psi}_{22}^{(f)} + \rho^0 (1-\eta^0) \beta_{21}^{(m)} \partial_{\bar{1}} \bar{\psi}_{22}^{(m)} + \partial_{\bar{2}} \bar{\bar{\Sigma}}_{21} = \rho^0 \ddot{\bar{u}}_1 \tag{5.28}$$

$$\rho^0 \bar{\gamma}_{12} \partial_{\bar{1}\bar{1}} \bar{u}_2 + \rho^0 \bar{\gamma}_{32} \partial_{\bar{3}\bar{3}} \bar{u}_2 + \rho^0 \eta^0 \beta_{12}^{(f)} \partial_{\bar{1}} \bar{\psi}_{21}^{(f)} + \rho^0 \eta^0 \beta_{32}^{(f)} \partial_{\bar{3}} \bar{\psi}_{23}^{(f)}$$
$$+ \rho^0 (1-\eta^0) \beta_{12}^{(m)} \partial_{\bar{1}} \bar{\psi}_{21}^{(m)} + \rho^0 (1-\eta^0) \beta_{32}^{(m)} \partial_{\bar{3}} \bar{\psi}_{23}^{(m)} + \partial_{\bar{2}} \bar{\Sigma}_{22} = \rho^0 \ddot{\bar{u}}_2 \tag{5.29}$$

$$\rho^0 \bar{\gamma}_{13} \partial_{\bar{1}\bar{1}} \bar{u}_3 + \rho^0 \bar{\gamma}_{33} \partial_{\bar{3}\bar{3}} \bar{u}_3 + \rho^0 \bar{\alpha} \partial_{\bar{1}\bar{3}} \bar{u}_1 + \rho^0 \eta^0 \beta_{23}^{(f)} \partial_{\bar{3}} \bar{\psi}_{22}^{(f)} + \rho^0 (1-\eta^0) \beta_{23}^{(m)} \partial_{\bar{3}} \bar{\psi}_{22}^{(m)} + \partial_{\bar{2}} \bar{\Sigma}_{23} = \rho^0 \ddot{\bar{u}}_3 \tag{5.30}$$

$$I_f^0 \gamma_{11}^{(f)} \partial_{\bar{1}\bar{1}} \bar{\psi}_{21}^{(f)} + I_f^0 \gamma_{31}^{(f)} \partial_{\bar{3}\bar{3}} \bar{\psi}_{21}^{(f)} + I_f^0 \alpha_{(f)} \partial_{\bar{1}\bar{3}} \bar{\psi}_{23}^{(f)} + \bar{\Sigma}_{21} - \rho^0 \beta_{12}^{(f)}) \partial_{\bar{1}} \bar{u}_2 - \rho_f^0 \gamma_{21}^{(f)} \bar{\psi}_{21}^{(f)} = I_f^0 \ddot{\bar{\psi}}_{21}^{(f)} \tag{5.31}$$

$$I_f^0 \gamma_{12}^{(f)} \partial_{\bar{1}\bar{1}} \bar{\psi}_{22}^{(f)} + I_f^0 \gamma_{32}^{(f)}) \partial_{\bar{3}\bar{3}} \bar{\psi}_{22}^{(f)} + \bar{\Sigma}_{22} - \rho^0 \beta_{21}^{(f)} \partial_{\bar{1}} \bar{u}_1 - \rho^0 \beta_{23}^{(f)} \partial_{\bar{3}} \bar{u}_3 - \rho_f^0 \gamma_{22}^{(f)} \bar{\psi}_{22}^{(f)} = I_f^0 \ddot{\bar{\psi}}_{22}^{(f)} \tag{5.32}$$

$$I_f^0 \gamma_{13}^{(f)} \partial_{\bar{1}\bar{1}} \bar{\psi}_{23}^{(f)} + I_f^0 \gamma_{33}^{(f)} \partial_{\bar{3}\bar{3}} \bar{\psi}_{23}^{(f)} + I_f^0 \alpha_{(f)} \partial_{\bar{1}\bar{3}} \bar{\psi}_{21}^{(f)} + \bar{\Sigma}_{23} - \rho^0 \beta_{32}^{(f)} \partial_{\bar{3}} \bar{u}_2 - \rho_f^0 \gamma_{23}^{(f)} \bar{\psi}_{23}^{(f)} = I_f^0 \ddot{\bar{\psi}}_{23}^{(f)} \tag{5.33}$$

$$I_m^0 \gamma_{11}^{(m)} \partial_{\bar{1}\bar{1}} \bar{\psi}_{21}^{(m)} + I_m^0 \gamma_{31}^{(m)} \partial_{\bar{3}\bar{3}} \bar{\psi}_{21}^{(m)} + I_m^0 \alpha_{(m)} \partial_{\bar{1}\bar{3}} \bar{\psi}_{23}^{(m)} + \bar{\Sigma}_{21} - \rho^0 \beta_{12}^{(m)} \partial_{\bar{1}} \bar{u}_2 - \rho_m^0 \gamma_{21}^{(m)} \bar{\psi}_{21}^{(m)} = I_f^0 \ddot{\bar{\psi}}_{21}^{(m)} \tag{5.34}$$

$$I_m^0 \gamma_{12}^{(m)} \partial_{\bar{1}\bar{1}} \bar{\psi}_{22}^{(m)} + I_m^0 \gamma_{32}^{(m)} \partial_{\bar{3}\bar{3}} \bar{\psi}_{22}^{(m)} + \bar{\Sigma}_{22} - \rho^0 \beta_{21}^{(m)} \partial_{\bar{1}} \bar{u}_1 - \rho^0 \beta_{23}^{(m)} \partial_{\bar{3}} \bar{u}_3 - \rho_m^0 \gamma_{22}^{(m)} \bar{\psi}_{22}^{(m)} = I_m^0 \ddot{\bar{\psi}}_{22}^{(m)} \tag{5.35}$$

$$I_m^0 \gamma_{13}^{(m)} \partial_{\bar{1}\bar{1}} \bar{\psi}_{23}^{(m)} + I_m^0 \gamma_{33}^{(m)} \partial_{\bar{3}\bar{3}} \bar{\psi}_{23}^{(m)} + I_m^0 \alpha_{(m)} \partial_{\bar{1}\bar{3}} \bar{\psi}_{21}^{(m)} + \bar{\Sigma}_{23} - \rho^0 \beta_{32}^{(m)} \partial_{\bar{3}} \bar{u}_2 - \rho_m^0 \gamma_{23}^{(m)} \bar{\psi}_{23}^{(m)} = I_m^0 \ddot{\bar{\psi}}_{23}^{(m)} \tag{5.36}$$

and

$$\partial_{\bar{2}} \bar{u}_l = \eta^0 \bar{\psi}_{2l}^{l} + (1-\eta^0) \bar{\psi}_{2l}^{(m)}. \tag{5.37}$$

In equations (5.28–5.37),

$$\partial_{\bar{l}} = \frac{\partial(\ )}{\partial \bar{x}_l^0}, \qquad \partial_{\bar{l}\bar{n}} = \frac{\partial^2(\ )}{\partial \bar{x}_l^0 \partial \bar{x}_n^0},$$

where $\bar{x}_l^0$ is defined by (5.16). We have also introduced

$$\bar{\psi}_{2l}^{(f)} = \frac{1}{\lambda_2^{(f)}} \hat{\psi}_{2l}^{(f)}, \qquad \bar{\psi}_{2l}^{(m)} = \frac{1}{\lambda_2^{(m)}} \hat{\psi}_{2l}^{(m)}, \qquad \bar{\Sigma}_{2l} = \frac{1}{\lambda_1 \lambda_3} \hat{\Sigma}_{2l}, \qquad I_f^0 = \frac{1}{12} \rho_f^0 (d_f^0)^2,$$
$$I_m^0 = \frac{1}{12} \rho_m^0 (d_m^0)^2. \tag{5.38}$$

The coefficients appearing in (5.28–5.36) can be written as:

$$
\left.
\begin{aligned}
\bar{\gamma}_{11} &= \frac{1}{\rho^0}\left[\eta^0 \frac{\partial t_1^{(f)}}{\partial \log \lambda_1} + (1-\eta^0)\frac{\partial t_1^{(m)}}{\partial \log \lambda_1}\right] = \frac{1}{\rho^0}\frac{\partial \bar{t}_1}{\partial \log \lambda_1} \\
\bar{\gamma}_{12} &= \frac{1}{\rho^0}\left[(\lambda_1)^2\eta^0 \frac{(t_1^{(f)}-t_2)}{(\lambda_1)^2-(\lambda_2^{(f)})^2} + (\lambda_1)^2(1-\eta^0)\frac{(t_1^{(m)}-t_2)}{(\lambda_1)^2-(\lambda_2^{(m)})^2}\right] \\
\bar{\gamma}_{13} &= \frac{(\lambda_1)^2}{\rho^0}\frac{(\bar{t}_1-\bar{t}_3)}{(\lambda_1)^2-(\lambda_3)^2} \\
\bar{\gamma}_{31} &= \frac{(\lambda_3)^2}{\rho^0}\frac{(\bar{t}_1-\bar{t}_3)}{(\lambda_1)^2-(\lambda_3)^2} \\
\bar{\gamma}_{32} &= \frac{1}{\rho^0}\left[(\lambda_3)^2\eta^0 \frac{(t_3^{(f)}-t_2)}{(\lambda_3)^2-(\lambda_2^{(f)})^2} + (\lambda_3)^2(1-\eta^0)\frac{(t_3^{(m)}-t_2)}{(\lambda_3)^2-(\lambda_2^{(m)})^2}\right] \\
\bar{\gamma}_{33} &= \frac{1}{\rho^0}\frac{\partial \bar{t}_3}{\partial \log \lambda_3} \\
\bar{\alpha} &= \frac{1}{2}\left[\frac{1}{\rho^0}\frac{\partial \bar{t}_1}{\partial \log \lambda_3} + \frac{1}{\rho^0}\frac{\partial \bar{t}_3}{\partial \log \lambda_1} + \bar{\gamma}_{13} + \bar{\gamma}_{31}\right] \\
\beta_{12}^{(f)} &= \frac{1}{\rho^0}\left[\frac{(\lambda_2^{(f)})^2 t_1^{(f)} - (\lambda_1)^2 t_2}{(\lambda_1)^2-(\lambda_2^{(f)})^2}\right] \\
\beta_{21}^{(f)} &= \frac{1}{\rho^0}\left.\frac{\partial(\lambda_1 t_2)}{\partial \lambda_1}\right|_{\lambda_2^{(f)}} = \frac{1}{\rho^0}\frac{\partial(\lambda_2^{(f)} t_1^{(f)})}{\partial \lambda_2^{(f)}} \\
\beta_{32}^{(f)} &= \frac{1}{\rho^0}\left[\frac{(\lambda_2^{(f)})^2 t_3^{(f)} - (\lambda_3)^2 t_2}{(\lambda_3)^2-(\lambda_2^{(f)})^2}\right] \\
\beta_{23}^{(f)} &= \frac{1}{\rho^0}\left.\frac{\partial(\lambda_3 t_2)}{\partial \lambda_3}\right|_{\lambda_2^{(f)}} = \frac{1}{\rho^0}\frac{\partial(\lambda_2^{(f)} t_3^{(f)})}{\partial \lambda_2^{(f)}} \\
\gamma_{11}^{(f)} &= \frac{1}{\rho_f^0}\frac{\partial t_1^{(f)}}{\partial \log \lambda_1} \\
\gamma_{12}^{(f)} &= \frac{(\lambda_1)^2}{\rho_f^0}\left[\frac{t_1^{(f)}-t_2}{(\lambda_1)^2-(\lambda_2^{(f)})^2}\right] \\
\gamma_{13}^{(f)} &= \frac{(\lambda_1)^2}{\rho_f^0}\left[\frac{t_1^{(f)}-t_3^{(f)})}{(\lambda_1)^2-(\lambda_3)^2}\right] \\
\gamma_{31}^{(f)} &= \frac{(\lambda_3)^2}{\rho_f^0}\left[\frac{t_1^{(f)}-t_3^{(f)}}{(\lambda_1)^2-(\lambda_3)^2}\right] \\
\gamma_{32}^{(f)} &= \frac{(\lambda_3)^2}{\rho_f^0}\left[\frac{t_3^{(f)}-t_2}{(\lambda_3)^2-(\lambda_2^{(f)})^2}\right] \\
\gamma_{33}^{(f)} &= \frac{1}{\rho_f^0}\frac{\partial t_3^{(f)}}{\partial \log \lambda_3}
\end{aligned}
\right\}
\quad (5.39)
$$

$$\left.\begin{aligned}
\bar{\alpha}_{(f)} &= \frac{1}{2}\left[\frac{1}{\rho_f^0}\frac{\partial t_1^{(f)}}{\partial \log \lambda_3}+\frac{1}{\rho_f^0}\frac{\partial t_3^{(f)}}{\partial \log \lambda_1}+\gamma_{13}^{(f)}+\gamma_{31}^{(f)}\right]\\
\gamma_{21}^{(f)} &= \frac{(\lambda_2^{(f)})^2}{\rho_f^0}\left[\frac{t_1^{(f)}-t_2}{(\lambda_1)^2-(\lambda_2^{(f)})^2}\right]\\
\gamma_{22}^{(f)} &= \frac{1}{\rho_f^0}\frac{\partial t_2}{\partial \log \lambda_2^{(f)}}\\
\gamma_{23}^{(f)} &= \frac{(\lambda_2^{(f)})^2}{\rho_f^0}\left[\frac{t_3^{(f)}-t_2}{(\lambda_3)^2-(\lambda_2^{(f)})^2}\right].
\end{aligned}\right\}$$

The coefficients $\beta_{12}^{(m)}$, etc., can be obtained by replacing (f) with (m) in the foregoing expressions.

6. DISPERSION RELATIONS IN STRESSED LAMINATED COMPOSITES

The system of equations (5.28)–(5.37) is now used to examine the propagation of small amplitude harmonic waves in a stressed laminated composite. We assume that the small superimposed deformation has the form:

$$(\bar{u}_l, \bar{\psi}_{2l}^{(f)}, \bar{\psi}_{2l}^{(m)}, \bar{\Sigma}_{2l}) = (a_l, b_{2l}^{(f)}, b_{2l}^{(m)}, \sigma_{2l})\exp[ik(n_i\bar{x}_i^0-ct)], \tag{6.1}$$

where a_l, $b_{2l}^{(f)}$, $b_{2l}^{(m)}$, σ_{2l} are constant amplitudes, k is the wavenumber, n_i the unit vector in the direction of propagation, and c the phase velocity. We will examine the following special cases:

Case 1

Longitudinal waves propagating in the x_1-direction ($n_1 = 1$, $n_2 = n_3 = 0$).

For waves of this type the nonvanishing field variables have the form

$$(\bar{u}_1, \bar{\psi}_{22}^{(f)}, \bar{\psi}_{22}^{(m)}, \bar{\Sigma}_{22}) = (a_1, b_{22}^{(f)}, b_{22}^{(m)}, \sigma_{22})\exp[ik(\bar{x}_1^0-ct)]. \tag{6.2}$$

Substitution of (6.2) in (5.28–5.37) yields a system of homogeneous equations for the amplitudes $a_1, b_{22}^{(f)}, b_{22}^{(m)}, \sigma_{22}$. By requiring that the determinant of the coefficients vanish, the following dispersion relation is obtained:

$$\begin{vmatrix} (c^2-\bar{\gamma}_{11}) & (\beta_{21}^{(f)}-\beta_{21}^{(m)}) \\ 12(\beta_{21}^{(f)}-\beta_{21}^{(m)}) & \bar{k}^2(c^2-\bar{\gamma}_{12})-12\bar{\gamma}_{22} \end{vmatrix} = 0, \tag{6.3}$$

where we have defined the nondimensional wavenumber $\bar{k}$

$$\bar{k} = k(d_f^0+d_m^0). \tag{6.4}$$

We have also introduced an effective coefficient

$$\bar{\gamma}_{22} = \frac{1}{\rho^0}\left(\frac{\rho_{(f)}^0\gamma_{22}^{(f)}}{\eta^0}+\frac{\rho_{(m)}^0\gamma_{22}^{(m)}}{(1-\eta^0)}\right). \tag{6.5}$$

In the limit $\bar{k} \to 0$, the wave speed c becomes

$$c^2 = \bar{\gamma}_{11} - \frac{(\beta_{21}^{(f)} - \beta_{21}^{(m)})^2}{\bar{\gamma}_{22}}. \tag{6.6}$$

Case 2

Vertically polarized shear waves propagating in the x_1-direction ($n_1 = 1, n_2 = n_3 = 0$).

For this type of wave motion the nonvanishing field variables have the form

$$(\bar{u}_2, \bar{\psi}_{21}^{(f)}, \bar{\psi}_{21}^{(m)}, \bar{\Sigma}_{21}) = (a_2, b_{21}^{(f)}, b_{21}^{(m)}, \sigma_{21}) \exp[ik(\bar{x}_1^0 - ct)]. \tag{6.7}$$

The dispersion relation is obtained as

$$\begin{vmatrix} (c^2 - \bar{\gamma}_{12}) & (\beta_{12}^{(f)} - \beta_{12}^{(m)}) \\ 12(\beta_{12}^{(f)} - \beta_{12}^{(m)}) & \bar{k}^2(c^2 - \bar{\gamma}_{11}) - 12\bar{\gamma}_{21} \end{vmatrix} = 0, \tag{6.8}$$

where $\bar{\gamma}_{21}$ is defined as

$$\bar{\gamma}_{21} = \frac{1}{\rho^0}\left[\frac{\rho_f^0 \gamma_{21}^{(f)}}{\eta^0} + \frac{\rho_m^0 \gamma_{21}^{(m)}}{(1-\eta^0)}\right]. \tag{6.9}$$

In the limit $\bar{k} \to 0$, c reduces to

$$c^2 = \bar{\gamma}_{12} - \frac{(\beta_{12}^{(f)} - \beta_{12}^{(m)})^2}{\bar{\gamma}_{21}}. \tag{6.10}$$

Case 3

Horizontally polarized shear waves propagating in the x_1-direction ($n_1 = 1, n_2 = n_3 = 0$).

The nonvanishing field variables have the form

$$(\bar{u}_3, \bar{\psi}_{23}^{(f)}, \bar{\psi}_{23}^{(m)}, \bar{\Sigma}_{23}) = (a_3, b_{23}^{(f)}, b_{23}^{(m)}, \sigma_{23}) \exp[ik(\bar{x}_1^0 - ct)]. \tag{6.11}$$

In this case the equations of motion uncouple and we obtain a symmetric mode ($\bar{u}^3$) with constant phase velocity

$$c^2 = \bar{\gamma}_{13} \tag{6.12}$$

and an antisymmetric mode ($\bar{\psi}_{23}^{(f)}, \bar{\psi}_{23}^{(m)}, \bar{\Sigma}_{23}$) with phase velocity

$$c^2 = \bar{\gamma}_{13} + 12\frac{\bar{\gamma}_{23}}{\bar{k}^2}, \tag{6.13}$$

where

$$\bar{\gamma}_{23} = \frac{1}{\rho^0}\left[\frac{\rho_f^0 \gamma_{23}^{(f)}}{\eta^0} + \frac{\rho_m^0 \gamma_{23}^{(m)}}{(1-\eta^0)}\right]. \tag{6.14}$$

In the limit $\bar{k} \to 0$, $c^2 \to \infty$. If we introduce the frequency $\omega = ck$, (6.13) can be rewritten as

$$\omega^2 = \bar{\gamma}_{13} k^2 + \frac{12\bar{\gamma}_{23}}{(d_f^0 + d_m^0)^2}. \tag{6.15}$$

As $k \to 0$ we obtain the cut-off frequency

$$\omega^2 = \frac{12\bar{\gamma}_{23}}{(d_f^0 + d_m^0)^2}. \tag{6.16}$$

Case 4

Longitudinal waves propagating in the x_2-direction ($n_2 = 1$, $n_1 = n_3 = 0$).

In this case the nonvanishing field variables have the form

$$(\bar{u}_2, \bar{\psi}_{22}^{(f)}, \bar{\psi}_{22}^{(m)}, \bar{\Sigma}_{22}) = (a_2, b_{22}^{(f)}, b_{22}^{(m)}, \sigma_{22})\exp[ik(\bar{x}_2^0 - ct)]. \tag{6.17}$$

In the usual manner we obtain the dispersion relation

$$\begin{vmatrix} \rho_f^0[\gamma_{22}^{(f)} - \frac{1}{12}(\eta^0)^2\bar{k}^2c^2] & \rho_0 c^2 \\ \rho_0(\bar{\gamma}_{22} - \bar{k}^2c^2)\eta^0(1-\eta^0) & \rho_m^0[\gamma_{22}^{(m)} - \frac{1}{12}(1-\eta^0)^2\bar{k}^2c^2] \end{vmatrix} = 0. \tag{6.18}$$

In the limit $\bar{k} \to 0$, the phase velocity c becomes

$$c^2 = \frac{\rho_f^0\rho_m^0\gamma_{22}^{(f)}\gamma_{22}^{(m)}}{(\rho_0)^2\bar{\gamma}_{22}\eta^0(1-\eta^0)}. \tag{6.19}$$

Case 5

Transverse shear waves propagating in the x_2-direction ($n_2 = 1$, $n_1 = n_3 = 0$).

We consider a transverse shear wave traveling normal to the layering with its amplitude in the x_1-direction. It should be noted that the phase velocity of a shear wave propagating in the x_2-direction with amplitude in the x_3-direction can be obtained from the following results by replacing the subscript 1 by 3. The general solution of a transverse shear wave propagating normal to the layering is the sum of these two solutions. If the displacement is in the x_1-direction, the wave is described by the following field variables:

$$(\bar{u}_1, \bar{\psi}_{21}^{(f)}, \bar{\psi}_{21}^{(m)}, \bar{\Sigma}_{21}) = (a_1, b_{21}^{(f)}, b_{21}^{(m)}, \sigma_{21})\exp[ik(\bar{x}_2^0 - ct)]. \tag{6.20}$$

The dispersion relation is

$$\begin{vmatrix} \rho_f^0[\gamma_{21}^{(f)} - \frac{1}{12}(\eta^0)^2\bar{k}^2c^2] & \rho_0 c^2 \\ \rho_0(\bar{\gamma}_{21} - \bar{k}^2c^2)\eta^0(1-\eta^0) & \rho_m^0[\gamma_{21}^{(m)} - \frac{1}{12}(1-\eta^0)^2\bar{k}^2c^2] \end{vmatrix} = 0. \tag{6.21}$$

In the limit $\bar{k} \to 0$, we obtain

$$c^2 = \frac{\rho_f^0\rho_m^0\gamma_{21}^{(f)}\gamma_{21}^{(m)}}{(\rho^0)2\bar{\gamma}_{21}\eta^0(1-\eta^0)}. \tag{6.22}$$

7. CONCLUSIONS

In this paper we have presented the kinematics, dynamics and constitutive equations for a homogeneous continuum model of a laminated nonlinear elastic composite. The governing equations were obtained by a procedure similar to the one introduced for the linear case in Ref. [1], and further extended in Refs. [2] and [3]. It was assumed that in the continuum model the kinematics of a laminated composite can be described by the gross motion of the composite and by two local motions of the reinforcing layers and the matrix layers, respectively. The local motions are related to the gross motion by a constraint condition which represents the continuity of the displacement at the interfaces of the layers. In the continuum model, the dynamical balance laws consist of the gross balance of linear momentum and the balance of moment of linear momentum of the reinforcing layers and of the matrix layers. In the balance laws, the interface stress vector is introduced as an independent dynamical quantity which is to be determined by solving the balance laws. The theory is completed by formulating the constitutive equations for the stress resultants and moments of stress for elastic laminae. As shown in Ref. [3] for the linear case, by the present approach the constitutive equations can be written without difficulty for viscoelastic or more general material behavior of the layers.

The theory was used to investigate the propagation of small amplitude time-harmonic waves in a prestressed laminated composite. These waves are dispersive and the nature of the dispersion relations is similar to those obtained for a linear material, the major difference being that the coefficients depend on the state of deformation.

APPENDIX

Exact solution for the case of uniform deformation of the layers

The problem of a layered composite material which is in a state of uniform static deformation in each layer can be solved rigorously. In this appendix we will present the solution and show that it agrees with the static solution of the approximate theory presented in Section 4.

The geometry of the layering is shown in Fig. 1. Within the kth pair of layers the balance laws in terms of the Piola stress tensor are

$$\begin{aligned} \partial_K T^{(fk)}_{Kl} &= 0 \qquad -\tfrac{1}{2}d_f \le X_2 - X^{(fk)}_2 \le \tfrac{1}{2}d_f \\ \partial_K T^{(mk)}_{Kl} &= 0 \qquad -\tfrac{1}{2}d_m \le X_2 - X^{(fk)}_2 \le \tfrac{1}{2}d_m. \end{aligned} \tag{A.1}$$

We assume that the deformation within each layer is uniform, i.e.

$$\begin{aligned} x^{(fk)}_l(X_K) &= A^{(fk)}_{Kl} X_K + b^{(fk)}_l \qquad -\tfrac{1}{2}d_f \le X_2 - X^{(fk)}_2 \le \tfrac{1}{2}d_f \\ x^{(mk)}_l(X_K) &= A^{(mk)}_{Kl} X_K + b^{(mk)}_l \qquad -\tfrac{1}{2}d_m \le X_2 - X^{(mk)}_2 \le \tfrac{1}{2}d_m, \end{aligned} \tag{A.2}$$

where $A^{(fk)}_{Kl}$, $b^{(fk)}_l$, $A^{(mk)}_{Kl}$ and $b^{(mk)}_l$ are constants within a layer, but may vary from layer to layer. From equations (4.1) and (4.2) we conclude that the Piola stresses for the kth pair of layers are constant within the layers:

$$\begin{aligned} T^{(fk)}_{Kl} &= T^{(f)}_{Kl}(A^{(fk)}_{Kl}) \\ T^{(mk)}_{Kl} &= T^{(m)}_{Kl}(A^{(mk)}_{Kl}), \end{aligned} \tag{A.3}$$

where $T^{(f)}_{Kl}(\ )$ and $T^{(m)}_{Kl}(\ )$ are the response functions of the reinforcing and matrix layers, respectively. From (A.3) it is easily seen that the balance laws (A.1) are identically satisfied. We thus need only to satisfy continuity of the displacement and the stress vector at each interface. The continuity of the displacement at the interface between the kth reinforcing and matrix layers yields

$$x^{(fk)}_l(X_1, X^{(fk)}_2 - \tfrac{1}{2}d_f, X_3) = x^{(mk)}_l(X_1, X^{(mk)}_2 + \tfrac{1}{2}d_m, X_3). \tag{A.4}$$

At the interface between the $(k-1)$th reinforcing layer and the kth matrix layer we have

$$x_l^{(fk-1)}(X_1, X_2^{(fk-1)}+\tfrac{1}{2}d_f, X_3) = x_l^{(mk)}(X_1, X_2^{(mk)}-\tfrac{1}{2}d_m, X_3). \tag{A.5}$$

Since the stresses are constant in each layer, the continuity of the components of the stress vector at these interfaces can be written as

$$T_{2l}^{(fk)} = T_{2l}^{(mk)} \tag{A.6}$$

$$T_{2l}^{(fk-1)} = T_{2l}^{(mk)}. \tag{A.7}$$

Substituting (A.2) in (A.4) and (A.5), one sees that since X_1 and X_3 are arbitrary,

$$A_{\alpha l}^{(fk)} = A_{\alpha l}^{(mk)} \tag{A.8}$$

and

$$A_{\alpha l}^{(mk)} = A_{\alpha l}^{(fk-1)}, \tag{A.9}$$

where $\alpha = 1, 3$. By induction on k we have

$$A_{\alpha l}^{(fk)} = A_{\alpha l}^{(mk)} = A_{\alpha l}^{(fr)} = A_{\alpha l}^{(mr)} = A_{\alpha l} \tag{A.10}$$

for all (k, r). Combining (A.6) and (A.7), and using (A.10), we have

$$T_{2l}^{(f)}(A_{\alpha l}, A_{2l}^{(fk)}) = T_{2l}^{(f)}(A_{\alpha l}, A_{2l}^{(fk-1)}), \tag{A.11}$$

which implies that†

$$A_{2l}^{(fk)} = A_{2l}^{(fk-1)} = \ldots = A_{2l}^{(fr)} \equiv B_{2l}^{(f)}. \tag{A.12}$$

Similarly, it follows that

$$A_{2l}^{(mk)} = A_{2l}^{(mk-1)} = \ldots = A_{2l}^{(mr)} \equiv B_{2l}^{(m)}. \tag{A.13}$$

The constants $B_{2l}^{(f)}$ and $B_{2l}^{(m)}$ are related by

$$T_{2n}^{(f)}(A_{\alpha l}, B_{2l}^{(f)}) = T_{2n}^{(m)}(A_{\alpha l}, B_{2l}^{(m)}). \tag{A.14}$$

We can now write (A.2) in the form

$$\begin{aligned} x_l^{(fk)}(X_K) &= A_{\alpha l}X_\alpha + B_{2l}^{(f)}X_2 + b_l^{(fk)} \\ x_l^{(mk)}(X_K) &= A_{\alpha l}X_\alpha + B_{2l}^{(m)}X_2 + b_l^{(mk)}. \end{aligned} \tag{A.15}$$

The continuity conditions (A.4) and (A.6) then reduce to

$$b_l^{(fk)} + B_{2l}^{(f)}(X_2^{(fk)} - \tfrac{1}{2}d_f) = b_l^{(mk)} + B_{2l}^{(m)}(X_2^{(mk)} + \tfrac{1}{2}d_m) \tag{A.16}$$

$$b_l^{(fk-1)} + B_{2l}^{(f)}(X_2^{(fk-1)} + \tfrac{1}{2}d_f) = b_l^{(mk)} + B_{2l}^{(m)}(X_2^{(mk)} - \tfrac{1}{2}d_m). \tag{A.17}$$

Subtracting (A.17) from (A.16), we obtain

$$\begin{aligned} b_l^{(fk)} + B_{2l}^{(f)}X_2^{(fk)} - b_l^{(fk-1)} - B_{2l}^{(f)}X_2^{(fk-1)} &= d_f B_{2l}^{(f)} + d_m B_{2l}^{(m)} = (d_f + d_m)A_{2l} \\ &= (X_2^{(fk)} - X_2^{(fk-1)})A_{2l}, \end{aligned} \tag{A.18}$$

where we have defined

$$A_{2l} = \eta B_{2l}^{(f)} + (1-\eta)B_{2l}^{(m)}, \tag{A.19}$$

† We assume that the response functions $T_{2l}^{(f)}(\ )$ are one to one.

and where $\eta = d_f/(d_f+d_m)$. By induction (A.18) yields

$$b_l^{(fk)}+B_{2l}^{(f)}X_2^{(fk)}-b_l^{(fr)}-B_{2l}^{(f)}X_2^{(fr)} = (X_2^{(fk)}-X_2^{(fr)})A_{2l}. \tag{A.20}$$

By choosing $r = 0$ as a reference, we obtain

$$b_l^{(fk)}+B_{2l}^{(f)}X_2^{(fk)} = b_l^{(f0)}+B_{2l}^{(f)}X_2^{(f0)}+(X_2^{(fk)}-X_2^{(f0)})A_{2l}. \tag{A.21}$$

Similarly for the matrix, it can be shown that

$$b_l^{(mk)}+B_{2l}^{(m)}X_2^{(mk)} = b_l^{(m0)}+B_{2l}^{(m)}X_2^{(m0)}+(X_2^{(mk)}-X_2^{(m0)})A_{2l}. \tag{A.22}$$

Substituting (A.21) and (A.22) into (A.16) gives

$$b_l^{(f0)}+B_{2l}^{(f)}X_2^{(f0)}-A_{2l}X_2^{(f0)} = b_l^{(m0)}+B_{2l}^{(m)}X_2^{(m0)}-A_{2l}X_2^{(m0)} \equiv b_l. \tag{A.23}$$

By combining (A.21–A.23) with (A.15), the deformation in the kth pair of layers reduces to

$$\begin{aligned} x_l^{(fk)}(X_K) &= A_{\alpha l}X_\alpha+A_{2l}X_2^{(fk)}+b_l+B_{2l}^{(f)}(X_2-X_2^{(fk)}) \\ x_l^{(mk)}(X_K) &= A_{\alpha l}X_\alpha+A_{2l}X_2^{(mk)}+b_l+B_{2l}^{(m)}(X_2-X_2^{(mk)}), \end{aligned} \tag{A.24}$$

where $A_{\alpha l}$, $B_{2l}^{(f)}$, $B_{2l}^{(m)}$ satisfy (A.14) and A_{2l} is defined by (A.19). It is easily seen that we can write (A.24) in the form:

$$\begin{aligned} x_l^{(fk)}(X_K) &= \bar{x}_l(X_1, X_2^{(fk)}, X_3)+B_{2l}^{(f)}(X_2-X_2^{(fk)}) \\ x_l^{(mk)}(X_K) &= \bar{x}_l(X_1, X_2^{(mk)}, X_3)+B_{2l}^{(m)}(X_2-X_2^{(mk)}), \end{aligned} \tag{A.25}$$

where $\bar{x}_l(X_K)$ is given by

$$\bar{x}_l(X_K) = A_{\alpha l}X_\alpha+A_{2l}X_2+b_l, \tag{A.26}$$

which can be identified with the gross displacement. Since the stresses are constant within each layer, this solution is identical to the one presented in Section 5.

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Абстракт—Выводится приближенная нелинейная теория для описания механического поведения слоистых материалов, составленных из переменных слоев двух однородных материалов, подверженных большим деформациям. Теория основана на разложениях движения, состоящцх из двух членов, по толщине недеформированных слоев. Даются законы кинематики и равновесия, а также составляются определяющие уравнения для упругого поведения слоистых материалов. Приводится подробно определяющие уравнения для случая возмущения с малой амплитудой, наложенного на большую статическую деформацию. Используется система уравнений для исследования распространения гармонических волн в предварительно напряженных слоистых материалах.